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Geometrical entropies. The extended entropy

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Abstract. By taking into account a geometrical interpretation of the measurement process [1,2], we define a set of measures of uncertainty. These measures will be called *geometrical entropies*. The amount of information is defined by considering the metric structure in the probability space. Shannon-von Neumann entropy is a particular element of this set. We show the incompatibility between this element and the concept of variance as a measure of the statistical fluctuations. When the probability space is endowed with the generalized statistical distance proposed in reference [3], we obtain the *extended entropy*. This element, which belongs to the set of geometrical entropies, is fully compatible with the concept of variance. Shannon-von Neumann entropy is recovered as an approximation of the extended entropy. The behavior of both entropies is compared in the case of a particle in a square-well potential.

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1 Introduction

Let us consider a random M-dimensional variable x, $x_1, ..., x_M$ being the discrete set of the M possible mutually exclusive outcomes when performing N random independent experiments. The probability distribution $p(x) = (p_1, ..., p_M)$ for the possible outcomes of measurements of x obeys to a multinomial function [1,4,5]. The uncertainty when determining p_i is given by the variance

$$\sigma_i(p_i) = \frac{p_i(1-p_i)}{N} \cdot \tag{1}$$

This quantity provides a measure of the statistical fluctuations in a random experiment [6,7].

Let X denote an observable associated with the physical variable x, $|\chi_i\rangle$ $(i = 1, ..., M)$ being the corresponding set of eigenfunctions with eigenvalues x_i. For a state described by a density matrix $\rho = \sum_{i=1}^{M} \lambda_i |\varphi_i\rangle \langle\varphi_i|$ the probabilities for the M possible outcomes are given by

$$p_i = \langle\chi_i, \rho\chi_i\rangle \,. \tag{2}$$

These probabilities are the relevant (macroscopic) variables which describe our physical system. From a microscopic point of view, the probabilities appear as the statistical expectation values of the set of orthogonal projectors $\xi_i = |\chi_i\rangle \langle\chi_i|$ [2]. Observable ξ_i is associated with the number of occurrences of x_i when performing N measurements of x. Expression (1) provides the variance of ξ_i.

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The uncertainty contained in $p(x)$ is coincidental with the amount of additional information which is required in order to specify the value of x. Thus, variance (1) is a measure of the amount of information gained when we obtain the value x_i in a measure.

Theories of probability and mathematical statistics provide different measures for expressing the uncertainty. The description of an observable as a random variable makes it possible to use these measures for expressing the uncertainty in quantum physics. All the results obtained in this paper for quantum probability distributions can be easily translated to classical probability distributions. In the framework of information theory [8–11] Shannon entropy (or information entropy),

$$S^{(S)}(p) = -\sum_{i=1}^{M} p_i \log p_i, \tag{3}$$

provides a quantitative measure of the uncertainty associated with the probability distribution $p(x)$. When eigenvalues λ_i of a certain density matrix ρ are replaced by probabilities in expression (3) we obtain the von Neumann entropy [12] $S^{(S)}(\rho) = -\mathrm{Tr}\,\rho \ln \rho = -\sum_{i=1}^{M} \lambda_i \ln \lambda_i$. It measures the amount of uncertainty contained within the density operator [2,13].

Let X, Y be non-commuting observables representing two physical variables x and y. It has been pointed out that the standard formulation of the uncertainty principle, expressed in terms of variances, does not properly express the quantum uncertainty principle [14,15]. In this situation the entropic relations

$$S^{(S)}(x) + S^{(S)}(y) \geq S_{xy}, \tag{4}$$

provide stronger re-formulations of the Heisenberg uncertainty principle [15–19]. In equation (4) S_{xy} is a positive constant.

This paper is organized as follows: in Section 2 we propose some properties of any measure of uncertainty. Information geometry [1,2,20,21] directly relates distance and uncertainty. Taking into account this relation, we define in Section 3 a set of "geometrical uncertainties". From this set we obtain a family of "geometrical entropies". It is shown in Section 4 that Shannon (von Neumann) entropy belongs to this family. We demonstrate the incompatibility between this entropy and the concept of variance. We propose the extended entropy, which also belongs to the set of geometrical entropies, as a measure of the uncertainty. Some formal properties of the extended entropy are given in Section 5. Section 6 is devoted to apply the previous results to the case of a particle in a square well potential. The paper finishes with some concluding remarks.

2 Intuitive properties of the uncertainty

A M-dimensional random variable x can be characterized by its probabilistic scheme $E_p = \{E_i; p_i; x_i\}$, $i = 1, ..., M$. In this scheme $E_1, ..., E_M$ represent the random events (states in a quantum system), quantities $p_1, ..., p_M$ give their probabilities, and $x_1, ..., x_M$ are the values ascribed to the random events [22]. Let S be the total uncertainty contained in a probabilistic scheme. Our aim is to find a function $s_i = s_i(E_i)$ providing the contribution to S due to the ith event . We begin proposing some intuitive properties of function s_i.

Taking into account the independence among the events, we can state that quantity s_i must be independent of probabilities p_j $(i \neq j)$, thus

– **s.1.** $s_i \equiv s_i(E_i) = s_i(p_i)$. $s_i \equiv s_i(E_i) = s_i(p_i)$.

Total uncertainty should be obtained by adding the contributions s_i of all the M possible events, *i.e.*,

– **s.2.** $S(p) = S(p_1, \cdots, p_M) = \sum_{i=1}^{M} s_i(p_i)$.

The scheme with the most uncertainty is the one with equally likely outcomes, $p_i = 1/M$ for all i. Moreover, the more alternative outcomes the random experiment has, the larger its uncertainty. From these facts, we have

– **s.3.** Function $S_M = \sum_{i=1}^{M} s_i\left(\frac{1}{M}\right)$ is a monotonically increasing function of M.

The following properties will be related with variance (1). The extreme values of this function are $\sigma_i^{\max} = \sigma_i(1/2) = 1/4N$ (maximum statistical fluctuations) and $\sigma_i^{\min} = \sigma_i(0) = \sigma_i(1) = 0$ (no statistical fluctuations) respectively. Its inverse function is defined by

$$p_i(\sigma_i) \equiv \sigma_i^{-1}(\sigma_i) = \begin{cases} p_i^-, & \text{if } p_i \in \left[0, \frac{1}{2}\right] \\ p_i^+, & \text{if } p_i \in \left[\frac{1}{2}, 1\right] \end{cases}, \tag{5}$$

with $p_i^{\pm} = \frac{1}{2}\left(1 \pm \sqrt{1 - 4N\sigma_i}\right)$. From this, we can define

$$s_i(\sigma_i) = \begin{cases} s_i^-(\sigma_i), & \text{if } p_i \in \left[0, \frac{1}{2}\right] \\ s_i^+(\sigma_i), & \text{if } p_i \in \left[\frac{1}{2}, 1\right], \end{cases} \tag{6}$$

where

$$s_i^{\pm}(\sigma_i) = s_i(p_i^{\pm}). \tag{7}$$

When measuring statistical fluctuations, it is expected s_i and variance σ_i be compatible quantities. This compatibility suggests us the following intuitive properties:

– **s.4.** For a given value of N, $s_i(\sigma_i)$ is a monotonically increasing function of σ_i,

and

– **s.5.** $s_i^{\min} \equiv s_i(0) = s_i(1) = 0$ (No statistical fluctuations), and $s_i^{\max} \equiv s_i(\frac{1}{2})$ (maximum statistical fluctuations).

In the last property s_i is considered as a function over p_i.

The larger the difference of p_i from $1/2$, the lesser the statistical fluctuations of the event E_i. Decrease in fluctuations should be coincidental when the cases $p_i^{(1)} = 1/2 + \Delta p_i$, and $p_i^{(2)} = 1/2 - \Delta p_i$ are considered. On the other hand, it is expected that the decreasing rate of the amount of uncertainty should be equal in both cases. In this situation the first derivative of s_i satisfies

– **s.6.** $s_i'(p_i) > 0$, if $p_i \in [0, \frac{1}{2})$, $s_i'(p_i) < 0$, if $p_i \in (\frac{1}{2}, 1]$,

and

– **s.7.** $s_i'(\frac{1}{2} + \Delta p_i) = -s_i'(\frac{1}{2} - \Delta p_i)$, for all $\Delta p_i \in [0, \frac{1}{2}]$,

with $s_i'(p_i) = \mathrm{d}s(p_i)/\mathrm{d}p_i$. From **s.5** to **s.7**, we obtain the following property:

– **s.8.** $s_i(\frac{1}{2} + \Delta p_i) = s_i(\frac{1}{2} - \Delta p_i)$, for all $\Delta p_i \in [0, \frac{1}{2}]$,

which allows us to write

$$s_i^-(\sigma_i) = s_i^+(\sigma_i), \quad \forall \sigma_i \in \left[0, \frac{1}{4N}\right]. \tag{8}$$

Properties **s.4** to **s.8** are required in order to obtain the expected compatibility between variance (1) and function s_i.

3 Geometrical uncertainties and geometrical entropies

Let $P = \{p(x)\}$ be the set of multinomial probability distributions $p(x)$ over the random variable x. These distributions are parameterized by the M-dimensional real vector parameter $p = (p_1, ..., p_M)$. P can be treated as a statistical model and it is known as the multinomial statistical model [1]. When $p(x)$ is sufficiently smooth in $p_1, ..., p_M$, the statistical model P forms a M-dimensional manifold embedded in the set of all the possible probability distributions, where $p_1, ..., p_M$ play the role of a coordinate system [1,20,21]. Mutual relations of distributions are then understandable as geometrical properties of the manifold. The question is, what is the natural geometric structure to be introduced in a manifold consisting of a statistical model? The answer to this problem is given by information geometry [1,20,21]. This field studies the geometrical structures of the manifolds of probability distributions. An introduction to information geometry is given in reference [21]. Reference [20] reviews the geometry of the manifolds of statistical models. In reference [1] the information geometry is throughly analyzed.

As we will see in this section, information geometry relates in a natural way geometry and uncertainty (or information). The results provided in this work are based in this relation. The same idea underlies in the work by Balian *et al.* where a geometrical theory of statistical physics is proposed in terms of the Riemannian geometry [2].

When the inner product of two vectors belonging to the tangent space is defined, the manifold P is called a Riemannian space. In this case, the natural geometrical structure to be introduced in the manifold of probability distributions is given by the positive definite Riemannian metric [1–3,6,20,21,23–26]

$$\mathrm{d}s^2 = \sum_{i=1}^{M}\sum_{j=1}^{M} g_{ij}\mathrm{d}p_i\mathrm{d}p_j. \tag{9}$$

It has been demonstrated [1,20,21] that the inner product is naturally defined by the covariance of two random variables. Especially, when the inner product of two vectors belonging to the natural basis of the tangent space is considered, matrix $\{g_{ij}\}$ in (9) is an important quantity known as the Fisher information [1,20]. Term g_{ii} is a measure of the amount of information due to the event E_i [27,28]. It was Rao [25] who first proposed the Riemannian structure by using the Fisher information matrix. It is well known that the only invariant Riemannian metric is given by the Fisher information [1,20,21]. This is called the information metric.

When information metric is considered, the statistical meaning of distance (9) is elucidated by the Cramer-Rao theorem [1]. According to this theorem, the lowest variance in an estimation of p_i when the remaining probabilities p_j $(i \neq j)$ are unknown is given by g^{ii}, $\{g^{ij}\}$ being the inverse of the Fisher information matrix. This result is formally stated by the following inequality:

$$V_{ii} \geq g^{ii}, \tag{10}$$

where V_{ii} is the variance (uncertainty) associated with p_i. The lower bound in this equation is a particular form of the Cramer-Rao inequality. Expression (10) relates uncertainty and distance and it has been extended to quantum mechanics [23,29–31].

The above exposed results show how information geometry provides a natural relation between uncertainty and distance. This relation suggests us that functions s_i and coefficients g^{ii} should be related quantities, *i.e.*,

$$s_i \equiv s_i(g^{ii}) = s_i(p_i). \tag{11}$$

The second equality in (11) is a consequence of property **s.1** and it allows us to state the functional dependence $g^{ii} = g^{ii}(p_i)$.

From relation (11) and property **s.2** we obtain

$$S = \sum_{i=1}^{M} s_i = \sum_{i=1}^{M} s_i(g^{ii}). \tag{12}$$

This expression points out, as expected from the results provided by information geometry, how information is obtained from the metric structure defined in the probability space. With definition (11) uncertainty acquires a clear geometrical interpretation. This result is in good agreement with similar ideas proposed by other authors [2,32–35].

Given a metric structure $\{g^{ij}\}$, each possible distribution of functions $s = (s_i, ..., s_M)$ generates a different quantity S in (12). Thus we have the set of measures of uncertainty

$$U_g = \left\{ S_g(s) = \sum_{i=1}^{M} s_i(g^{ii}) \ \ \forall s \right\}. \tag{13}$$

On the other hand, by considering the possible distributions $g = (g^{11}, ..., g^{MM})$ which it is possible to define in a probability space, we obtain the set of sets

$$U = \left\{ U_g \ \forall g / g^{ii} = g^{ii}(p_i) \right\}. \tag{14}$$

We will refer to U as set of "geometrical uncertainties". The conceptual validity of each measure of uncertainty $S \subset U_g$ depends on the behavior, concerning to properties **s.1** to **s.8**, of function s_i.

Concavity is an expected property of function s_i [36]. Thus, given a metric tensor, we can take the concave function

$$s_i^{\ln}(g^{ii}) = -g^{ii} \ln g^{ii}. \tag{15}$$

This function generates the following element in U_g:

$$S_g(s^{\ln}) = \sum_{i=1}^{M} s_i^{\ln}(g^{ii}) = -\sum_{i=1}^{M} g^{ii} \ln g^{ii}, \tag{16}$$

with $s^{\ln} = (s_1^{\ln}, ..., s_M^{\ln})$. When all the possible Riemannian metrics are considered, expression (16) generates in (14) the family of measures of uncertainty

$$S^{\ln} = \left\{ S_g(s^{\ln}) \ \forall g / g^{ii} = g^{ii}(p_i) \right\} \subset U. \quad (17)$$

If we consider the functional dependence upon g^{ii}, quantities (16) exhibit the same formal behavior than the usual definition of entropy (3) when this quantity is considered as a function on p_i. By taking into account this formal relation, we will refer to the family (17) as "geometrical entropies". Geometrical entropies will allow us to introduce in a natural way the metric properties of the probability space in the entropic uncertainty relations (4).

4 Shannon-von Neumann entropy and extended entropy

The metric properties of a probability space are due to the statistical fluctuations in a finite sequence of measurements [6,24–26,30–32]. In this way, the components of the metric tensor are completely given by the uncertainties (variances). After drawing N samples from a probability distribution, one can estimate the probabilities as the observed frequencies. As it has been pointed out in Section 1, the probability distribution for the frequencies is given by a multinomial distribution with variances given by (1). When N is large enough, the local limit theorem provides an asymptotic approximation to the multinomial distribution. When this approximation is considered, the probability for the frequencies is proportional to a Gaussian distribution with variances [5,6,23]

$$\sigma_i^{(n)} = \frac{p_i}{N} \cdot \quad (18)$$

These uncertainties generate the "natural metric" (see, for instance, Ref. [6]),

$$g_n^{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ p_i & \text{if} \quad i = j \end{cases}. \quad (19)$$

This tensor usually provides the metric in the manifold P of multinomial distributions [1,2,6,23,25,28,37].

By taking $g^{ii} = g_n^{ii} = p_i$ in (17), the natural metric generates the element $S_{g_n}(s^{\ln})$ in the family of geometrical entropies. In this case contribution (15) due to the ith event will be given by

$$s_i^{(S)}(p_i) \equiv s_i^{\ln}(g_n^{ii}) = -p_i \ln p_i. \quad (20)$$

From this and expression (12) we can see that Shannon entropy (3) is the member of the geometrical entropies generated by the natural metric, *i.e.*, $S_{g_n} = S^{(S)}$. This result can be applied to the von Neumann entropy when eigenvalues λ_i are replaced by probabilities p_i in the previous expressions. In this way, the Shannon (von Neumann) entropy is in a normal manner associated with the natural metric (19). This conclusion is in agreement with the result provided in reference [2], in which it is shown how the natural metric is generated by the Shannon (von Neumann) entropy.

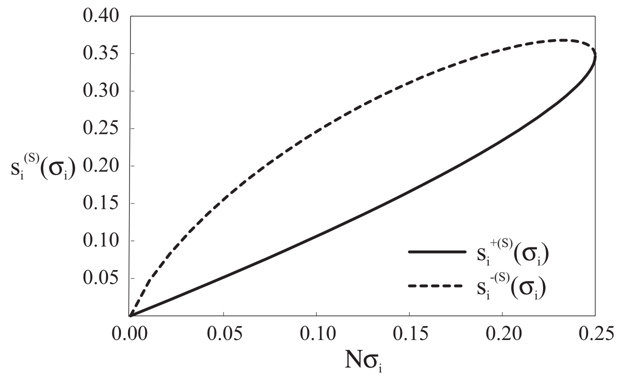

Fig. 1. Function $s_i^{(S)}(\sigma_i)$ plotted against $N\sigma_i$. As it is shown in equation (6), the values of this function are given by $s_i^{-(S)}(\sigma_i)$ when $p_i \in [0, 1/2]$, and by $s_i^{+(S)}(\sigma_i)$ when $p_i \in [1/2, 1]$. In order to compute the amount of information, we have token the natural logarithm. Thus, the units of $s_i^{(S)}(\sigma_i)$ are given in Nats.

We now analyze the behavior of Shannon entropy regarding properties proposed in Section 2. Quantities $s_i^{(S)}$ and $S^{(S)}$ satisfy properties **s.1** to **s.3** [8]. The maximum of function $s_i^{(S)}(\sigma_i)$, as defined in (6), occurs at $\sigma_i = (e-1)/Ne^2$, below $1/4N$ for all $N \geq 1$. From this, one conclude that this function does not satisfy **s.4**. By taking into account definition (7) we have (see Fig. 1)

$$s_i^{-(S)}(\sigma_i) \geq s_i^{+(S)}(\sigma_i), \quad \forall\, \sigma_i \in \left[0, \frac{1}{4N}\right]. \quad (21)$$

This inequality reveals an unexpected result for $s_i^{(S)}$: it is a bivaluated function on σ_i.

The second part of property **s.5** is not satisfied by function $s_i^{(S)}(p_i)$: this quantity reaches its maximum value at $p_i = 1/e$, with $s_i^{\max(S)} = s_i^{(S)}(1/e) = 1/e$ (see Fig. 2). According to this, the maximum amount of uncertainty attached to the ith event is obtained when its corresponding probability is $1/e$. If we think in the probabilistic scheme defined by a coin or dice, this result is not an intuitive one. It can be verified that function $s_i^{(S)}$ does not satisfy properties **s.6** to **s.8**. Inequality (21) points out how the information (von Neumann) entropy does not verify condition (8). In view of these results, we can conclude that the standard definition of entropy is not compatible with variance (1) as a measure of statistical fluctuations.

As it has been pointed out, the natural metric is generated by variances (18), which are recovered as an approximation of variances σ_i ($\sigma_i^{(n)} = \lim_{p_i \to 0} \sigma_i$). Thus, when the natural metric is introduced in the probability space, the metric properties are due to approximation (18) and not to the true variances (1). The above mentioned mutual incompatibility between Shannon (von Neumann) entropy and variance is a consequence of this fact.

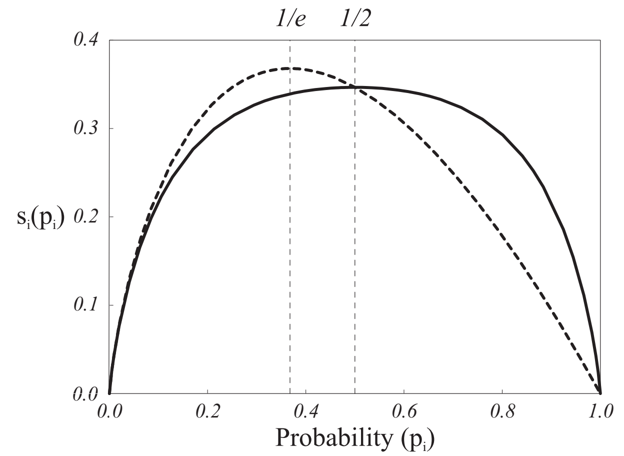

Fig. 2. Quantity $s_i(p_i)$ in the case of the Shannon (von Neumann) entropy (dashed line), and in the case of the extended entropy (solid line). The units of s_i are given in Nats.

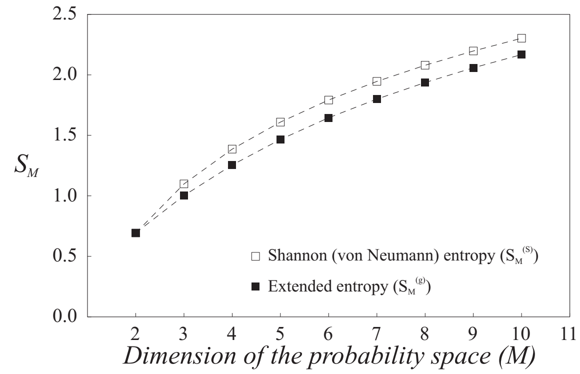

Fig. 3. Function S_M (in Nats) providing the maximum uncertainty in a M-dimensional probability space. The results predicted by the Shannon (von Neumann) entropy, $S_M^{(S)}$, are compared with those obtained by using the extended entropy, $S_M^{(g)}$.

In reference [3] an asymptotic approximation to the multinomial distribution is proposed. This approximation maintains the usual definition of variances (1) and it endows to the probability space with the "generalized statistical distance". The corresponding contravariant metric tensor is

$$g_g^{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ p_i(1-p_i) & \text{if} \quad i = j \end{cases}. \tag{22}$$

With this distance the metric properties of the probability space are due to the true variances (1).

If we take $g^{ii} = g_g^{ii}$ in (15) the contribution to the total uncertainty due to the event E_i will be given by

$$s_i^{(g)}(p_i) \equiv s_i^{\ln}(g_g^{ii}) = -p_i(1-p_i)\ln[p_i(1-p_i)]. \tag{23}$$

In this case, the element S_g in (16) generated by the generalized statistical distance can be written as

$$S^{(g)} \equiv S_{g_g}(s^{\ln}) = \sum_{i=1}^{M} s_i^{(g)}(p_i) = -\sum_{i=1}^{M} p_i(1-p_i)\ln p_i(1-p_i). \tag{24}$$

Quantities (23, 24) satisfy properties **s.1** and **s.2**. Property **s.3** is also satisfied: the maximum of $S^{(g)}$ is reached for the most random scheme ($p_i = 1/M$) and quantity $S_M^{(g)} = \sum_{i=1}^{M} s_i^{(g)}(1/M) = [(M-1)/M]\ln[M^2/(M-1)]$ is a monotonically increasing function of M (see Fig. 3). From expressions (1, 23), we obtain

$$s_i^{(g)}(\sigma_i) = -N\sigma_i \ln N\sigma_i, \tag{25}$$

which is a monotonically increasing function on σ_i. In this way, $s_i^{(g)}$ satisfies property **s.4**. It is immediately seen that this quantity verifies properties **s.5** to **s.8** (see Fig. 2).

The measure of uncertainty (24) satisfies all the properties listed in Section 2. Thus, quantity $S^{(g)}$ is fully compatible with variance (1).

By introducing expression (25) in (12) quantity $S^{(g)}$ can be rewritten as $S^{(g)} = -\sum_{i=1}^{M} N\sigma_i \ln N\sigma_i$. This result points out how this measure of uncertainty depends on the knowledge of variances while the specification of Shannon (von Neumann) entropy only requires the knowledge of probabilities. We will call to quantity $S^{(g)}$ "extended entropy". Using expression (1) when computing the diagonal terms in (22), one finds $g_g^{ii}(\sigma_i) = N\sigma_i$. In this way, for a fixed number of trials N, quantity g_g^{ii} provides us with a direct measure of the variance, as it was expected by considering inequality (10). This fact points out how the extended entropy is fully compatible with the geometrical interpretation of the measurement process. In a similar way, by introducing $p_i(\sigma_i)$, as defined in (5), in expression (19) we obtain

$$g_n^{ii}(\sigma_i) = \begin{cases} \frac{1}{2}\left(1 - \sqrt{1-4N\sigma_i}\right), \text{ if } p_i \in \left[0, \frac{1}{2}\right] \\ \frac{1}{2}\left(1 + \sqrt{1-4N\sigma_i}\right), \text{ if } p_i \in \left[\frac{1}{2}, 1\right] \end{cases}. \tag{26}$$

Thus the diagonal terms of the natural metric does not provide us with a direct measure of the variance. This result shows how standard entropy exhibits serious inconsistencies with the geometrical interpretation of the measurement process.

The maximum value of the Shannon entropy is $S_M^{(S)} = \ln M$. By comparing this value with the maximum, $S_M^{(g)}$, obtained in the case of the extended entropy we have inequality $S_M^{(g)} \leq S_M^{(S)}$ for all integer M. Thus, in any probability space, with $M > 2$, endowed with the extended entropy, the maximum uncertainty is lesser (see Fig. 3).

In the limit $p_i \to 0$ quantities (18) can be considered as approximations to the variances (1). From this, it becomes obvious that the natural metric will be recovered as an approximation to the generalized statistical distance:

$$g_n^{ii} = \lim_{p_i \to 0} g_g^{ii}. \quad (27)$$

By considering this result, we finally obtain

$$S^{(S)} = \lim_{p_i \to 0} S^{(g)}. \quad (28)$$

Condition $p_i \to 0$ is satisfied for the neighboring states to that with equally outcomes for large enough number M of possible outcomes. If we take into account that the larger M, the larger the uncertainty, expression (28) tells us that Shannon entropy is recovered as an approximation to the extended entropy when large enough values of the uncertainty are considered.

By replacing the eigenvalues λ_i by probabilities p_i in expression (24), we have that the extended entropy associated with the density operator ρ is

$$S^{(g)}(\rho) = -\mathrm{Tr}\left[\rho(1-\rho)\ln\rho(1-\rho)\right] = -\sum_{i=1}^{M} \lambda_i(1-\lambda_i)\ln\lambda_i(1-\lambda_i). \quad (29)$$

5 Some formal properties of the extended entropy

In this section we will demonstrate some formal properties of the extended entropy.

– **p.1.** Positivity. $S^{(g)}(\rho) \geq 0$, and $S^{(g)}(\rho) = 0$ if, and only if, ρ is a pure state ($\lambda_i = 1$ for some i).

Thus, when the system is described by a pure state there exist measurements (those relative to a basis containing $|\varphi\rangle$, if $\rho = |\varphi\rangle\langle\varphi|$), whose result can be predicted with absolute certainty.

– **p.2.** Equiprobability. $S^{(g)}(\rho) \leq S_M^{(g)}(\rho)$ and the maximum value is achieved when $\lambda_i = 1/M$ for all i. This upper limit is a concave function of M and diverges as $M \to \infty$ (see Fig. 3).

From this property we obtain that the probabilistic scheme with more uncertainty is that with equally likely eigenvectors.

Function $s_i^{(g)}(x_i) = -x_i(1-x_i)\ln x_i(1-x_i)$ is concave in the interval $[0,1]$. Let $\rho = a\rho_a + (1-a)\rho_b$ ($a \in [0,1]$) be the quantum state obtained by fitting together two mixed states ρ_a and ρ_b. From concavity of $s_i^{(g)}$ extended entropy (29) satisfies [36]

$$\begin{aligned} S^{(g)}(\rho) &= \sum_{i=1}^{M} s_i^{(g)}(\langle\varphi_i|\rho|\varphi_i\rangle) \\ &\geq a\sum_{i=1}^{M} s_i^{(g)}(\langle\varphi_i|\rho_a|\varphi_i\rangle) + (1-a)\sum_{i=1}^{M} s_i^{(g)}(\langle\varphi_i|\rho_b|\varphi_i\rangle) \\ &\geq a\sum_{i=1}^{M}\left\langle\varphi_i\left|s_i^{(g)}(\rho_a)\right|\varphi_i\right\rangle + (1-a)\sum_{i=1}^{M}\left\langle\varphi_i\left|s_i^{(g)}(\rho_b)\right|\varphi_i\right\rangle \\ &= aS^{(g)}(\rho_a) + (1-a)S^{(g)}(\rho_b), \end{aligned} \quad (30)$$

and we have

– **p.3.** Concavity. $S^{(g)}(\rho) \geq aS^{(g)}(\rho_a) + (1-a)S^{(g)}(\rho_b)$.

Concavity formalizes a well known fact: uncertainty in an mixture always increases.

We now consider $\rho \in \mathrm{H}$ and $0 \in \mathrm{H}''$, H and H″ being Hilbert spaces. Density matrix $\rho' = \rho \oplus 0$, which belongs to the space $\mathrm{H}' = \mathrm{H} \oplus \mathrm{H}''$, satisfies

– **p.4.** Expansibility. $S^{(g)}(\rho') = S^{(g)}(\rho)$.

Extended entropy remains invariant when we add events with vanishing probabilities.

We assume that the time evolution of probabilities in a physical system obeys to the master equation [36]

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \sum_{j=1}^{M}\left(w_{ij}p_j - w_{ji}p_i\right), \quad (31)$$

w_{ij} being the transition probability, per unit time, from the jth state to the ith one. If detailed balance holds [38], *i.e.*, $w_{ij} = w_{ji}$, the time evolution of the extended entropy can be written as

$$\frac{\mathrm{d}S^{(g)}}{\mathrm{d}t} = -\sum_{i=1}^{M}\sum_{j=1}^{M} w_{ji}(p_j - p_i)(1-2p_i)\left[\ln p_i(1-p_i) + 1\right]. \quad (32)$$

If we interchange subindex i and j in this equation, by summing both results, we obtain

$$\frac{\mathrm{d}S^{(g)}}{\mathrm{d}t} = \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M} w_{ij}(p_i - p_j)f(p_i,p_j), \quad (33)$$

with $f(p_i,p_j) = (1-2p_i)\left[\ln p_i(1-p_i)+1\right] - (1-2p_j)\left[\ln p_j(1-p_j)+1\right]$. Function f satisfies $f(p_i,p_j) \geq 0$ if $p_i \geq p_j$, and $f(p_i,p_j) < 0$ if $p_i < p_j$ for all p_i, p_j such that $0 \leq p_i, p_j \leq 1$. From this property and expression (33), we can state

– **p.5.** Irreversibility (H-theorem). $\mathrm{d}S^{(g)}/\mathrm{d}t \geq 0$.

This property points out how the irreversibility associated with the second law of thermodynamics remains valid for the extended entropy.

From expression (20) we obtain the following relation between the standard and the extended entropy:

– **p.6.** Relation with the standard entropy. $S^{(g)}(\rho) = S^{(S)}(\rho) + S_\sigma^{(S)}(\rho) - S_c^{(S)}(\rho)$, with

$$S_\sigma^{(S)}(\rho) = \sum_{i=1}^{M} \lambda_i s_i^{(S)}(1-\lambda_i), \quad (34)$$

and

$$S_c^{(S)}(\rho) = \sum_{i=1}^{M} \lambda_i s_i^{(S)}(\lambda_i). \quad (35)$$

Let ρ_a and ρ_b be density operators, with eigenvalues μ_i ($i = 1, ..., L$) and η_j ($j = 1, ..., M$) respectively, representing two uncorrelated systems a and b. The composite system ab obtained by fitting together both individual systems will be described by the density matrix ρ_{ab} with eigenvalues

$$\lambda_k = \mu_i \eta_j \quad (1 \le i \le L,\, 1 \le j \le M,\, 1 \le k \le L \times M). \tag{36}$$

In this situation standard entropy is an additive quantity, *i.e.* information about the total system equals the sum of the information about its constituents. When eigenvalues (36) are introduced in definition (29), we obtain the extended entropy of the composite system,

$$S^{(g)}(\rho_{ab}) = S^{(g)}(\rho_a \otimes \rho_b) = -\sum_{i=1}^{L}\sum_{j=1}^{M} \mu_i\eta_j(1-\mu_i\eta_j)\ln \mu_i\eta_j(1-\mu_i\eta_j). \tag{37}$$

If we take into account the additivity of von Neumann entropy, by applying property **p.6** to (37), we obtain

– **p.7.** Pseudo-additivity. $S^{(g)}(\rho_a \otimes \rho_b) = S^{(g)}(\rho_a) + S^{(g)}(\rho_b) + S_\sigma(\rho_a \otimes \rho_b) - S_c(\rho_a \otimes \rho_b)$, with

$$S_{\sigma,c}(\rho_a \otimes \rho_b) = S^{(S)}_{\sigma,c}(\rho_a \otimes \rho_b) - S^{(S)}_{\sigma,c}(\rho_a) - S^{(S)}_{\sigma,c}(\rho_b), \tag{38}$$

$$S^{(S)}_\sigma(\rho_a \otimes \rho_b) = -\sum_{i=1}^{L}\sum_{j=1}^{M} \mu_i\eta_j(1-\mu_i\eta_j)\ln(1-\mu_i\eta_j), \tag{39}$$

and

$$S^{(S)}_c(\rho_a \otimes \rho_b) = -\sum_{i=1}^{L}\sum_{j=1}^{M} \mu_i^2\eta_j^2 \ln \mu_i\eta_j. \tag{40}$$

This property clearly points out how generalized entropy is a non-extensive quantity. $S^{(g)}(\rho_a \otimes \rho_b)$ can becomes larger or smaller than $S^{(g)}(\rho_a) + S^{(g)}(\rho_b)$ in a non trivial way. This behavior is checked in Figure 4. In this figure $S^{(g)}(\rho_a \otimes \rho_b)$ is plotted against $S^{(g)}(\rho_a) + S^{(g)}(\rho_b)$ for two pseudo-random sets of 1000 distributions μ_1, ..., μ_L, and η_1, ..., η_M when different values of L and M are considered.

Properties **p.1** to **p.5**, and **p.7** are those of a generalized entropy. These kind of entropies have been introduced in order to extend the Boltzmann-Gibbs thermodynamics by generalizing the concept of entropy to non-extensive physics [39–42].

We now analyze property **p.6** from an information-theoretical point of view. Quantity $S^{(S)}(p)$ is the amount of information gained when we know the probabilities in a random experiment. In an analogous way, it is expected to gain some additional information from knowledge of variances. From expression (1) the second term in the right-hand side of property **p.6** can be rewritten as

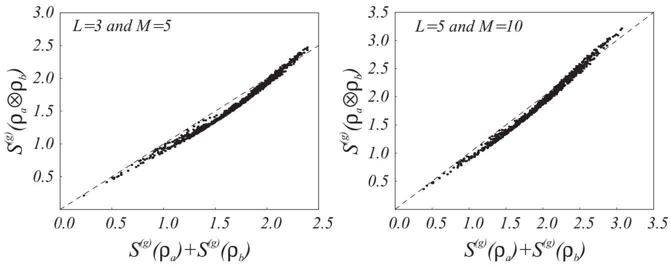

Fig. 4. Extended entropy for a composite system, $\rho_a \otimes \rho_b$, plotted against the sum of the extended entropies of the individual systems, ρ_a and ρ_b. In each graph, L and M are the dimensions of the Hilbert spaces associated with the density matrices ρ_a and ρ_b respectively. The results shown in this figure have been obtained by using two sets of 1000 pseudo-random distributions of eigenvalues, μ_1, ..., μ_L for ρ_a, and η_1, ..., η_M for ρ_b. The units in both axes are given in Nats.

$$S^{(S)}_\sigma(p) = \sum_{i=1}^{M} p_i s_i^{(S)}\left(\frac{N\sigma_i}{p_i}\right). \tag{41}$$

The dependence on σ_i suggests us that quantity $s_i^{(S)}(N\sigma_i/p_i)$ is the additional amount of information gained from the knowledge of the ith variance. Thus, expression (41) provides the average amount of information gained by the observer due to the knowledge of the M variances σ_i. Quantity $S^{(S)}(p) + S^{(S)}_\sigma(p)$ is the amount of information due to the knowledge of probabilities and variances.

As it is pointed out by equation (1), variances and probabilities are correlated variables and knowledge of p_i tends to reduce uncertainty in σ_i [43]. In this way, due to this correlation, a certain amount of information contained in $S^{(S)}(p) + S^{(S)}_\sigma(p)$ is not removed and it should be subtracted of this quantity. Term $S^{(S)}_c(p)$ in **p.6** contributes to the total uncertainty as a negative quantity and it can be associated with the amount of uncertainty contained in the correlation between variances and probabilities. Thus, in the context of information theory, property **p.6** allows us to interpret the extended entropy as the amount of uncertainty removed in a probabilistic scheme due to the knowledge of probabilities and variances minus that amount of uncertainty contained in the correlation between both variables. The amount of information associated with the correlation between variances and probabilities has been pointed out by different authors [13,43,44].

We now analyze the meaning of property **p.7**. From relation (36), property **p.6** allows us to write the extended entropy of the composite system $\rho_a \otimes \rho_b$ as

$$S^{(g)}(\rho_a \otimes \rho_b) = S^{(S)}(\rho_a \otimes \rho_b) + S^{(S)}_\sigma(\rho_a \otimes \rho_b) - S^{(S)}_c(\rho_a \otimes \rho_b), \tag{42}$$

with

$$S^{(S)}(\rho_a \otimes \rho_b) = -\sum_{i=1}^{L}\sum_{j=1}^{M} \mu_i\eta_j \,\ln\, \mu_i\eta_j$$
$$= S^{(S)}(\rho_a) + S^{(S)}(\rho_b), \qquad (43)$$

and $S^{(S)}_{\sigma,c}(\rho_a \otimes \rho_b)$ given by (39, 40). The additivity of von Neumann entropy (43) is a consequence of the trivial relation (36). In order to specify the extended entropy (42) we need to know probabilities $\mu_i\eta_j$ and their corresponding variances σ_{ij}. In contrast to the case of probabilities, variances of the composite system, σ_{ij}, are related with those of the individual systems, σ_i and σ_j, in a non-trivial way. From this fact, it is expected the extended entropy to be a non-additive quantity.

The information gained due to the knowledge of variances and correlations in a composite system is given by quantity $S^{(S)}_{\sigma}(\rho_a \otimes \rho_b) - S^{(S)}_{c}(\rho_a \otimes \rho_b)$. From the previous arguments, the non-additivity is an expected property of this quantity. Now we will formalize this fact. From (34) and property $\mathrm{Tr}\,\rho_a = \mathrm{Tr}\,\rho_b = 1$ we obtain

$$S^{(S)}_{\sigma}(\rho_a) + S^{(S)}_{\sigma}(\rho_b) = -\sum_{i=1}^{L}\sum_{j=1}^{M} \mu_i\eta_j(1-\mu_i)\ln(1-\mu_i)$$
$$-\sum_{i=1}^{L}\sum_{j=1}^{M} \mu_i\eta_j(1-\eta_j)\ln(1-\eta_j). \qquad (44)$$

If we take into account the trivial inequalities $1-\mu_i\eta_j \geq 1-\mu_i$, and $1-\mu_i\eta_j \geq 1-\eta_j$, the last expression, and definition (39) it can be shown that the following relation holds:

$$-S^{(S)}_{\sigma}(\rho_a \otimes \rho_b) + S^{(S)}_{\sigma}(\rho_a) + S^{(S)}_{\sigma}(\rho_b) \geq$$
$$\sum_{i=1}^{L}\sum_{j=1}^{M} \mu_i\eta_j(\mu_i+\eta_j-\mu_i\eta_j-1)\ln(1-\mu_i\eta_j). \qquad (45)$$

When restricted to the region $x, y \geq 0$, the maximum of function $h(x,y) = x+y-xy-1$ is achieved at the point $(x,y) = (1,1)$, with $h(1,1) = 0$. From this fact one easily verifies that condition $\mu_i+\eta_j-\mu_i\eta_j-1 \leq 0$ is satisfied in (45) and we have

$$S^{(S)}_{\sigma}(\rho_a \otimes \rho_b) \leq S^{(S)}_{\sigma}(\rho_a) + S^{(S)}_{\sigma}(\rho_b). \qquad (46)$$

This inequality points out how $S^{(S)}_{\sigma}(\rho_a \otimes \rho_b)$ in (42) is a sub-additive quantity, *i.e.*, the amount of information associated with the knowledge of variances of the individual constituents is greater than the amount of information provided by the knowledge of variances of the composite system. This result is an intuitive one: the knowledge of the individual variances allows us to obtain the variances of the composite system, however, it is not possible to recover variances σ_i and σ_j from variances σ_{ij}.

Applying definition (35) to the individual systems ρ_a, and ρ_b one has

$$S^{(S)}_{c}(\rho_a) + S^{(S)}_{c}(\rho_b) = -\sum_{i=1}^{L}\sum_{j=1}^{M} \mu_i^2\eta_j \ln \mu_i$$
$$-\sum_{i=1}^{L}\sum_{j=1}^{M} \mu_i\eta_j^2 \ln \eta_j. \qquad (47)$$

From this result and expression (40) we arrive at

$$S^{(S)}_{c}(\rho_a \otimes \rho_b) \leq S^{(S)}_{c}(\rho_a) + S^{(S)}_{c}(\rho_b), \qquad (48)$$

i.e., function (35) is also a sub-additive quantity.

Relations (46, 48) point out how the term $S^{(S)}_{\sigma} - S^{(S)}_{c}$ in (42) will not be, in general, an additive quantity. This non-additivity is the responsible of property **p.7**. From the previous analysis, it can be concluded that property **p.7** is a natural consequence of the contribution of variances and correlations to the definition of the extended entropy.

We now discuss another interpretation of the extended entropy. In a random experiment it is expected that the non-occurrence of an event will provide us with some additional information about the physical system. The probability of non-occurrence for the ith event is given by

$$q_i = 1 - p_i. \qquad (49)$$

In the context of the information theory the total uncertainty removed in a probabilistic scheme due to the non-occurrence of its events will be given by the corresponding Shannon entropy

$$S^{(S)}_{n}(p) = -\sum_{i=1}^{M}(1-p_i)\ln(1-p_i)$$
$$= -\sum_{i=1}^{M} q_i \ln q_i = S^{(S)}(q). \qquad (50)$$

The contribution to this quantity associated with the ith event is

$$s^{(S)}_{i}(q_i) = -q_i \ln q_i = -(1-p_i)\ln(1-p_i). \qquad (51)$$

When Shannon entropy is considered, expressions (49, 51) allow us to rewrite definition (34) in the form

$$S^{(S)}_{\sigma}(p) = \sum_{i=1}^{M} p_i s^{(S)}_{i}(q_i). \qquad (52)$$

In this way, the amount of information $S^{(S)}_{\sigma}$ associated with the knowledge of variances is coincidental with the average amount of information gained from the knowledge of probabilities of non-occurrence.

By using expressions (49, 51), the extended entropy (24) can be expressed as

$$S^{(g)}(p) = S^{(g)}_{y}(p) + S^{(g)}_{n}(p), \qquad (53)$$

with

$$S_y^{(g)}(p) = \sum_{i=1}^{M} q_i s_i^{(S)}(p_i), \tag{54}$$

and $S_n^{(g)}(p) \equiv S_\sigma^{(S)}(p)$. Quantity $S_y^{(g)}$ is the average of the contributions $s_i(p_i)$ when weighted by probabilities of non-occurrence. We can consider this quantity as the amount of uncertainty removed in a random experiment due to the knowledge of the probabilities associated with the occurrence of the different outcomes. In a similar way, quantity $S_n^{(g)}(p)$ can be associated with the amount of information gained due to the knowledge of probabilities of non-occurrence in a given probabilistic scheme. Thus, the information provided by the extended entropy takes into account the amount of information due to the probabilities of occurrence and that obtained from the knowledge of probabilities of non-occurrence. From an intuitive point of view, this is an expected result: if probabilities of non-occurrence provide information about the system, it seems a natural fact to consider them when computing the total information removed in a random experiment. In this sense, quantities $S_y^{(g)}$ and $S_n^{(g)}$ provide complementary informations. This behavior is a consequence of relation (49): a variation δp_i implies a complementary change $\delta q_i = -\delta p_i$.

When two quantum uncorrelated systems, ρ_a and ρ_b, are considered, definitions (52, 54) allow us to state the following inequalities:

$$S_y^{(g)}(\rho_a \otimes \rho_b) \geq S_y^{(g)}(\rho_a) + S_y^{(g)}(\rho_b), \tag{55}$$

and

$$S_n^{(g)}(\rho_a \otimes \rho_b) \leq S_n^{(g)}(\rho_a) + S_n^{(g)}(\rho_b). \tag{56}$$

These relations point out how terms $S_y^{(g)}$ and $S_n^{(g)}$ are super-additive and sub-additive quantities respectively. This result formally expresses the complementary behavior of both quantities.

Expression (53) provides an important interpretation for the extended entropy: it is the total amount of information gained by an observer in a random experiment by considering the uncertainty removed due to the knowledge of probabilities of occurrence and the information gained from the knowledge of probabilities of non-occurrence.

6 Entropies for a particle in a square well potential

A well known entropic uncertainty relation, which belongs to the set of inequalities (4), is [19]

$$S_{\mathbf{r}}^{(S)} + S_{\mathbf{k}}^{(S)} \geq m(1 + \ln \pi). \tag{57}$$

This inequality expresses the restrictions imposed by quantum theory on probability distributions of canonically conjugate variables, $\mathbf{r}$ and $\mathbf{k}$, in terms of the corresponding information theory entropies [19],

$$S_{\mathbf{r}}^{(S)} = \langle \ln \rho_{\mathbf{r}}(\mathbf{r}) \rangle = -\int \mathrm{d}^m \mathbf{r} \rho_{\mathbf{r}}(\mathbf{r}) \ln \rho_{\mathbf{r}}(\mathbf{r}), \tag{58}$$

$$S_{\mathbf{k}}^{(S)} = \langle \ln \rho_{\mathbf{k}}(\mathbf{k}) \rangle = -\int \mathrm{d}^m \mathbf{k} \rho_{\mathbf{k}}(\mathbf{k}) \ln \rho_{\mathbf{k}}(\mathbf{k}). \tag{59}$$

The position-space entropy, $S_{\mathbf{r}}^{(S)}$, measures the uncertainty in the localization of a particle in space. In the same way, momentum-space entropy, $S_{\mathbf{k}}^{(S)}$, provides the uncertainty in predicting the momentum of the particle. Functions $\rho_{\mathbf{r}}(\mathbf{r}) = |\psi_{\mathbf{r}}(\mathbf{r})|^2$ and $\rho_{\mathbf{k}}(\mathbf{k}) = |\psi_{\mathbf{k}}(\mathbf{k})|^2$ are probability densities in $m-$dimensional position and momentum spaces, $\psi_{\mathbf{r}}(\mathbf{r})$ and $\psi_{\mathbf{k}}(\mathbf{k})$ being the corresponding wave functions. The total information entropy is given by $S^{(S)} = S_{\mathbf{r}}^{(S)} + S_{\mathbf{k}}^{(S)}$.

Expressions (58, 59) provide the discrete von Neumann entropies when continuous random variables, $\mathbf{r}$ and $\mathbf{k}$, are considered. In a similar way, by applying definition (29), quantities

$$\begin{aligned} S_{\mathbf{r}}^{(g)} &= -\langle (1-\rho_{\mathbf{r}}) \ln[\rho_{\mathbf{r}}(1-\rho_{\mathbf{r}})] \rangle \\ &= -\int \mathrm{d}^m \mathbf{r} \rho_{\mathbf{r}}(1-\rho_{\mathbf{r}}) \ln[\rho_{\mathbf{r}}(1-\rho_{\mathbf{r}})], \end{aligned} \tag{60}$$

and

$$\begin{aligned} S_{\mathbf{k}}^{(g)} &= -\langle (1-\rho_{\mathbf{k}}) \ln[\rho_{\mathbf{k}}(1-\rho_{\mathbf{k}})] \rangle \\ &= -\int \mathrm{d}^m \mathbf{k} \rho_{\mathbf{k}}(1-\rho_{\mathbf{k}}) \ln[\rho_{\mathbf{k}}(1-\rho_{\mathbf{k}})], \end{aligned} \tag{61}$$

are the position and momentum extended entropies. The total extended entropy is $S^{(g)} = S_{\mathbf{r}}^{(g)} + S_{\mathbf{k}}^{(g)}$.

Let us consider a particle in a one-dimensional ($m = 1$) box with perfectly rigid and impenetrable walls located at points $x = \pm L/2$. When pure energy states are considered, this physical situation is described by the probability densities for position x and momentum k

$$\rho_x(x, n, L) = \begin{cases} \dfrac{2}{L} \cos^2 \dfrac{n\pi x}{L}, \text{ if } n \text{ odd} \\ \dfrac{2}{L} \sin^2 \dfrac{n\pi x}{L}, \text{ if } n \text{ reven}, \end{cases} \tag{62}$$

and

$$\rho_k(k, n, L) = \begin{cases} 4n^2 L\pi \dfrac{\cos^2(kL/2)}{(n^2\pi^2 - k^2L^2)^2}, \text{ if } n \text{ odd} \\ 4n^2 L\pi \dfrac{\sin^2(kL/2)}{(n^2\pi^2 - k^2L^2)^2}, \text{ if } n \text{ even}. \end{cases} \tag{63}$$

By introducing these expressions in definitions (58–61) we have computed the position, $S_x^{(S,g)}(n, L)$, and momentum, $S_k^{(S,g)}(n, L)$, entropies. The total entropies are $S^{(S,g)}(n, L) = S_x^{(S,g)}(L) + S_k^{(S,g)}(n, L)$. These entropies are shown in Figure 5 for $L = 4$. The results obtained when

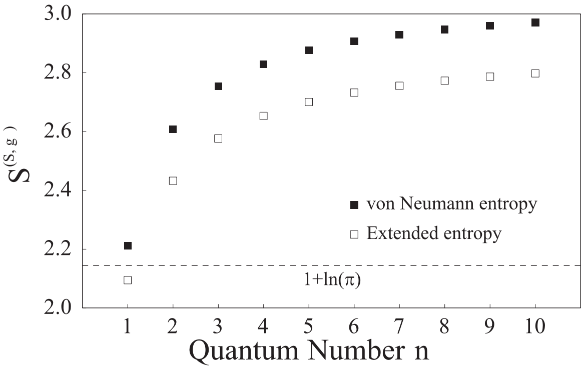

Fig. 5. Total (position plus momentum) von Neumann and extended entropies for a particle in a one-dimensional box computed as functions on the quantum number n. The size of the box is $L = 4$ arb. units. The dashed line shows the lower limit imposed by the entropic uncertainty relation (4). The units of entropy are Nats.

different values of L considered are qualitatively equal to those provided in this figure. Extended and von Neumann entropy satisfy inequality

$$S^{(g)}(n, L) < S^{(S)}(n, L). \quad (64)$$

It is shown in Figure 5 that total extended entropy can be lesser than the lower bound in the right-hand side of the entropic uncertainty relation (57). By replacing extended entropy by von Neumann entropy, inequality (64) allows us to obtain a stronger version of the entropic uncertainty relation (57).

7 Conclusions

We have described some general properties of a suitable measure of uncertainty, S, in a random experiment. Some of these properties, **s.4** to **s.8**, are stated by considering the expected compatibility between quantity S and the concept of variance when measuring statistical fluctuations.

From the geometrical interpretation of the measurement process provided by information geometry, and formally expressed by relation (10), we have defined the set U of geometrical uncertainties. The measures of uncertainty belonging to this set are generated by the contravariant metric tensors $\{g^{ii}\}$. In this way, the amount of information removed in a random experiment depends upon the metric structure generated by the statistical fluctuations in a finite sequence of measurements. This result provides a geometrical interpretation of the uncertainty and it allows us to introduce the metric properties of the probability space in the entropic uncertainty relations (4). The quantum uncertainty principle can be formulated in terms of geometrical quantities.

When the concave function $s_i^{\ln}(g^{ii}) = -g^{ii} \ln g^{ii}$ is introduced in the set of geometrical uncertainties, we obtain the family of geometrical entropies, $S^{\ln}$. Each element of this subset is generated by a different metric tensor. We have demonstrated that Shannon (von Neumann) entropy is the member of $S^{\ln}$ obtained when the probability space is endowed with the natural metric. This entropy does not satisfy properties **s.4** to **s.8** and it becomes incompatible with variances (1). This inconsistency arises from the fact that natural metric is obtained from an asymptotic approximation with variances given by (18), which differ from the true variances (1). As a consequence of this result we question the validity of the Shannon (von Neumann) entropy as a suitable measure of uncertainty in a random experiment.

In this situation we have used the generalized statistical distance (22). When the probability space is endowed with this distance, the metric properties are given by the true variances (1). The element of $S^{\ln}$ generated by the generalized statistical distance is the extended entropy $S^{(g)}$. We have demonstrated that this measure of uncertainty satisfies properties **s.1** to **s.8**. Thus, when measuring statistical fluctuations, extended entropy is fully compatible with variances (1). Extended entropy reproduces, in the $p_i \to 0$ limit, the standard Shannon entropy.

In the context of the information theory, extended entropy can be considered as the amount of information gained in a probabilistic scheme due to the knowledge of probabilities and variances minus the amount of uncertainty contained in the correlation between both magnitudes. In this way, extended entropy is associated with a second order statistics. On the other hand, extended entropy can be considered as the total amount of information, $S_y^{(g)} + S_n^{(g)}$, obtained by summing the information gained due to the knowledge of probabilities of occurrence and that provided by the knowledge of probabilities of non-occurrence. Quantities $S_y^{(g)}$ and $S_n^{(g)}$ provide complementary information. This result can be formally stated by considering that both functions are super-additive and sub-additive quantities respectively.

Extended entropy satisfies the most important properties of Shannon (von Neumann) entropy, except that of additivity, and all the properties of a generalized entropy, including pseudo-additivity [39–42] and the H-theorem. Taking into account the above exposed considerations, non-additivity becomes an expected property of S^g.

The case of a particle in a square well potential has been analyzed. From this analysis, it is shown that von Neumann and extended entropies exhibit a similar qualitative behavior. The amount of uncertainty provided by the extended entropy is always lesser than that obtained from the von Neumann entropy. This result allows us to obtain a stronger versions of the entropic uncertainty relations (4).

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